ON NONABELIAN p-GROUPS OF A GIVEN ORDER

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Abstract. In every nonabelian p-group G, possessing two cyclic subgroups X_i and X_j, the quotient group of G by X_i is isomorphic to the cyclic group X_j for $i, j \in \{1, 2\}$, $i \neq j$.
Moreover, if $p > 2$ and G is metacyclic, possessing a nonabelian section Y, of order p^3, then Y is a trivial subgroup of G.
This paper proposes solution to problem $8^0(1)$ of section 124 in [Ber.Y].

1. Introduction

Many of the basic properties of finite p-groups were proved by Burnside, Frobenius, Sylow and a host of other Mathematicians.

This paper purposes to present some of the input efforts in classifying the p-groups most especially, those with a cyclic subgroup of index p. Part of the work also leads to the computation of the number of subgroups of a given order in a metacyclic p-group.

2. Basic Definitions and Notations

* $B^x = x^{-1}Bx = \{b^x | b \in B\}$ for $x \in G$, $B \subseteq G$.
* $[a, b] = a^{-1}b^{-1}ab = a^{-1}a^b$. The commutator of elements a and b of a group G.
* $|W|$ is the cardinality of the set W. If G is a finite group, then $|G|$ is called the order of G.
* $\circ(x)$ is the order of an element x of G.
* $cl(G)$ is the nilpotence class of a p-group G. Here, there exists a series given by: $G = G_0 \geq G_1 \geq \cdots \geq G_n \geq G_{n+1} = \{e\}$. And we say that G is of class n. We then write $cl(G) = n > 1$.
* A p-group of maximal class is a nonabelian group G of order p^m, $m \geq 3$ with $cl(G) = m - 1 > 1$.
* Let $G = G_1 \times \cdots \times G_n$, $A \leq G$ and $a \in A$. Then $a = (a_1, \dots, a_n)$, where $a_i \in G_i \ \ \forall\ i$. Define a projection $\pi_i : A \longrightarrow G_i$, setting $\pi_i(a) = a_i$, all $a \in A$. Then, π_i is a homomorphism and $A_i = \pi_i(A)$ is an epimorphic image of A. This is called the section of A. Obviously, $A \leq A_1 \times \cdots \times A_n$.
* A section of a group G is an epimorphic image of some subgroups of G.
* $\Omega_n(G) = \langle x \in G | \circ(x) \leq p^n \rangle$
* $\mho_n(G) = \langle x^{p^n} | x \in G \rangle$
* $A \lhd B \Longrightarrow A$ is a nontrivial normal subgroup of G, where $H > \{1\}$.
* A p-group G is said to be homocyclic if it is of type $(p^n, p^n, \dots, p^n)$, $n > 0$.

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- $*$ Metacyclic Group: This is a group G in which both its commutator subgroup G' and the quotient group G/G' are cyclic. It has a cyclic normal subgroup L such that G/L is also cyclic.

3. Proof of the Results

The following Theorem was given in 1897 by Burnside, in his book: Theory of Groups of Finite Order (first edition).

Theorem 1 [see Ber. Y., Mar, H.]. Suppose that G is a nonabelian p-group, $|G| = p^{n+1}$ and $\langle x\rangle = A \subseteq G$, of index p. Then, one of the following groups is isomorphic to G.

(i) $R_{p^{n+1}} = \langle x, y | x^{p^n} = y^p = 1,\ y^{-1}xy = x^{1+p^{n-1}}\rangle$ where $n \geq 3$ if $p = 2$.

(ii) For $p = 2$

(a) The Dihedral Group given by

$$D_{2^{n+1}} = \langle x, y | x^{2^n} = y^2 = 1,\ yxy = x^{-1}\rangle$$

(b) The Generalized Quaternion Group given by

$$Q_{2^{n+1}} = \langle x, y | x^{2^n} = 1,\ y^2 = x^{2^{n-1}}, y^{-1}xy = x^{-1}\rangle$$

or

(c) The Semidihedral Group

$$SD_{2^{n+1}} = \langle x, y | x^{2^n} = y^2 = 1,\ \ yxy = x^{-1+2^{n-1}}\rangle;\ n > 2$$

Definition: Let G be a group. If G is nonabelian but all its proper subgroups are abelian, then G is said to be minimal nonabelian.

Proposition 1: [Ber, Y.]: Suppose that G is a metacyclic p-group containing a nonabelian subgroup V of order p^3. Then

(i) if $p = 2$, G is of maximal class

(ii) if $p > 2$, then $|G| = p^3 \Rightarrow G = V$.

Lemma 2 (see [Ber, Y.][Scot.]): Suppose that G is a nonabelian metacyclic p-group.

(i) If G is of order p^4 and exponent p^2 then G is minimal nonabelian. Moreso, if $p = 2$, then G is isomorphic to the group

$$H_2 = \langle x, y | x^4 = y^4 = 1,\ \ x^y = x^3\rangle,$$

where all subgroups of order 2 are characteristic in G. All maximal cyclic subgroups of G have order p^2.

(ii) By Proposition 1. If G has a nonabelian subgroup of order p^3, then it is of maximal class. To be specific, if $p > 2$, then $|G| = p^3$.

(iii) Let B be a normal subgroup of G and $p = 2$ such that G/B is nonabelian of order 2^3, then B is characteristic in G.

(iv) By Theorem 1 and Lemma 2, there are exactly four series of nonabelian 2-groups with cyclic subgroup of index 2:viz : R_{2^n}, D_{2^n}, Q_{2^n}, and SD_{2^n}.

Lemma 3 (O. Taussky). Suppose that G is a nonabelian 2-group and $|G : G'| = 4$. Then G is as in theorem 1 (ii(a),(b),(c)).

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Lemma 4 [Ber, Y.] Let G be a metacyclic 2-group with a nonabelian section of order 2^3. If G is not of maximal class, then

(i) There exists a normal subgroup $T \triangleleft G$ such that G/T is isomorphic to Q_{2^3} and $G/\mho_2(G)$ is isomorphic to H_2.

(ii) $\mho_1(G)$ has no nonabelian section of order 2^3.

Here comes the analysis of the first part of the assertion.
Let G be as in theorem 1.
If $B_i \subseteq G$, $i \in \{1, 2\}$, such that B_i is cyclic, we assert that

$$|G/B_i| = |B_j|, \ \ i \neq j$$

In doing this, it suffices to show that G/B_i is isomorphic to B_j, $i \neq j$, $i, j \in \{1, 2\}$.
Define a mapping as follows

$$\begin{aligned} f \quad &: (G/B_i) \longrightarrow B_j \quad i \neq j \\ &: xB_i \mapsto y \end{aligned} \tag{$*$}$$

We show that $(*)$ is (i) a monomorphism, (ii) an epimorphism.
Suppose that $|B_i| < |B_j|$, $i \neq j$. Then for all $x \in G$, there exists $y \in B_j$ such that $f(xB_i) = y$.
Also, if $f(x_1B_i) = f(x_2B_i) = y$.
Then,

$$x_1B_i = x_2B_i \tag{2}$$

And by cancellation law (see [Kur.], [Ros]) post multiply both sides of (2) by B_i^{-1} we have that

$$\begin{aligned} x_1B_iB_i^{-1} &= x_2B_iB_i^{-1} \\ \Rightarrow \ x_1 &= x_2. \end{aligned}$$

This confirms (i) and (ii) as stated above.
The case is similar for $|B_i| > |B_j|$, $i \neq j$, $i, j \in \{1, 2\}$ □

In dealing with the second aspect of the problem, the following items are very imperative:
Define the upper Ω-series:

$$\{1\} = \Omega_{(0)}(G) < \Omega_{(1)}(G) < \cdots < \Omega_{(s)}(G) < \cdots$$

of a p-group G as follows:
$\Omega_{(0)}(G) = \{1\}$
$\Omega_{(i+1)}(G)/\Omega_{(i)}(G) = \Omega_1(P/\Omega_{(i)}(G)), \ i = 0, 1, \ldots$
Clearly, $\Omega_i(G) \leq \Omega_{(i)}(G)$.
$\Omega_{(i)}(G) = \Omega_{(i+1)}(G)$ implies $\Omega_{(i)}(G) = G$.
But $\Omega_{(i)}(G) = \Omega_{(i+1)}(G)$ does not imply that $\Omega_i(G) = G$.
Now, set

$$\left|\Omega_{(i+1)}(G) : \Omega_{(i)}(G)\right| = p^{t_{i+1}}$$
$$|\mho_i(G) : \mho_{i+1}(G)| = p^{v_{i+1}}, \ \ i = 0, 1, \ldots$$

Definition A_1 **[Ber. Y.].**

(i) G is said to be upper pyramidal if $t_1 \geq t_2 \geq \cdots$

(ii) G is said to be lower pyramidal if $v_1 \geq v_2 \geq \cdots$.

Definition A_2**:** A p-group G is said to be generalized homocyclic if it satisfies the following conditions as noted in definition A_1.

(i) $t_1 = t_2 = \cdots = v_1 = v_2 = \cdots$

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(ii) $\Omega_{(i+1)}(G)/\Omega_{(i)}(G)$ are abelian for all nonegative integer i.
(iii) If D is a term of the upper or lower central series of G, then $D = \Omega_i(G)$ for some nonnegative integer i.

If G is a generalized homocylic group of exponent p^e, then $\Omega_{(i)}(G) = \Omega_i(G)$ and $\exp(\Omega_i(G)) = p^i \ \forall\, i \leq e$ [Ber, Y.].
Now, suppose that a metacyclic p-group, $p > 2$ has a nonabelian section Y, of order p^3. If $p = 2$ and G is not of maximal class, then the result is in harmony with Lemma 4 (i) and (ii).

On the other hand, if $p > 2$, then going by induction on $|G|$, assuming G is nonabelian. Then, by Lemma 3, if G is neither cyclic nor a 2-group of maximal class, then $\Omega_1(G) \cong Ep^2$.

By proposition 1, $|G| = p^2$.
$\Longrightarrow G = Y$ (its section)
$\Longrightarrow Y$ is a trivial subgroup of G.
If G is nonabelian of order p^4 and exponent p^2, it is not generalized homocyclic (see section 8 of [Ber, Y]). (We have that $t_1 = t_2 = v_1 = v_2 = 2$),
If $p > 2$, then G is metacyclic. So, $|G/G'| = p^3$ and $G' \neq \Omega_1(G)$ □.

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